10/13/21

Last time:

$D(\vec{p}) > 0$ and $f_{xx}(\vec{p}) > 0 \Rightarrow \vec{p}$ a local min "up like a cup"

$D(\vec{p}) > 0$ and $f_{xx}(\vec{p}) < 0 \Rightarrow \vec{p}$ a local max "down like a frown"

$D(\vec{p}) < 0 \Rightarrow p$ is a saddle point

1-* if $D(\vec{p}) = 0$ can't conclude anything!

2-* could use f_{yy} instead of f_{xx}

- ex: Classify CP using 2nd der test

critical point at $\nabla f = 0$

$f(x,y,z) = x^2 + xy + y^2 + y$

$\nabla f = \langle 2x + y, x + 2y + 1 \rangle$

$\nabla f = 0$ iff $2x + y = 0$, $y + 2y + 1 = 0$ $\quad \begin{cases} y = -2x \\ x + 2(-2x) + 1 = 0 \\ 3x - 1 = 0 \end{cases} \Rightarrow x = \frac{1}{3},\ y = \frac{-2}{3}$

$CP = \left(\frac{1}{3}, \frac{-2}{3}\right)$

- via second der test,

D_{xy}

$f_{xx}\left(\frac{1}{3}, \frac{-2}{3}\right) = 2 \quad f_{yy}\left(\frac{1}{3}, \frac{-2}{3}\right) = 2 \quad f_{xy} = 1$

$D_{x,y} = f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3$

classify pt.

at $P = \left(\frac{1}{3}, -\frac{2}{3}\right)$, $D(p) = 3 > 0$ + $f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$

$\rightarrow$ a local minimum point

$f\left(\frac{1}{3}, \frac{-2}{3}\right) = x^2 + xy + y^2 + y \ @ \left(\frac{1}{3}, \frac{-2}{3}\right) = \frac{1}{3}$

- ex: Classify critical points of $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9y$

critical points

$\nabla f = \langle 3x^2 - 6x, 3y^2 - 6y - 9 \rangle$

$\nabla f = 0 = x(x-2), (y-3)(y+1)$ $\quad$ ($y^2 - 2y - 3$)

$x = 0 \quad y = 3$

$x = 2 \quad y = -1 \Rightarrow CP: (0,3), (2,3), (0,-1), (2,-1)$

	$x=0$	$x=2$
$y=3$	$(0,3)$	$(2,3)$
$y=-1$	$(0,-1)$	$(2,-1)$

grid for visualization

- via second der test,

D_{xy}

$f_{xx} = \langle 6x - 6 \rangle \quad f_{yy} = \langle 6y - 6 \rangle \quad f_{xy} = 0$

$D_{xy} = f_{xx} \cdot f_{yy} - f_{xy}^2 = 6^2(x-1)(y-1)$

analyzing

@ $(0,3)$: $D_{0,3} < 0$ saddle point

@ $(0,-1)$: $D_{0,-1} > 0$ $\quad f_{xx}(0,-1) < 0$ so a local max $f(0,-1) = 13$

@ $(2,3)$: $D_{2,3} > 0$ $\quad f_{xx}(2,3) > 0$ so local min at $f(2,3) = -3$

@ $(2,-1)$: $D_{2,-1} < 0$ saddle point

* calculate values at min/max not saddle points

- **ex:** $f(x,y) = xy + e^{-xy}$

$\nabla f = \langle y - ye^{-xy},\ x - xe^{-xy} \rangle$

$\nabla f = 0$ iff $y(1-e^{-xy}) = 0$ | or $x(1-e^{-xy}) = 0$

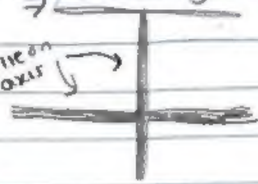
$y=0$ or $1-e^{-xy}=0$ | $x=0$ or $1-e^{-xy}=0$

iff $e^{-xy} = e^0$

$\nabla f = 0$ iff $y=0 \;||\; (x==0 \;||\; y==0) \Rightarrow x==0 \;||\; y==0$

$x=0 \;||\; (y==0 \;||\; x==0)$

CP lie on axis



$f_{xx} = y^2 e^{-xy}$ $f_{yy} = x^2 e^{-xy}$

$f_{xy} = (1-e^{-xy})(1-xy)$

$D(x,y) = (xy)^2 e^{-xy} - ((1-e^{-xy})(1-xy))^2$

$D(0,y) = 0 - ((0)(1-0))^2 = 0$
$D(x,0) = 0$
} when $D(x,y)$ we learn nothing!

{ ?: LaGrange Multipliers }

Goal: Build a method to systematically solve constrained optimization methods

{ Optimize $f(\vec{x})$ subject to $g_1(\vec{x}) = g_2(\vec{x}) = \dots g_N(\vec{x}) = 0$ ← constraints

- Observe: If we want extremes of F and want to live on level set $\vec{F}(\vec{x}) = c$ we really want critical points of $F(\vec{x})$, because $\nabla F = \nabla C = 0$

$F(\vec{x}, \lambda_1, \dots, \lambda_N) = f(\vec{x}) - \lambda_1 g_1(\vec{x}) - \dots - \lambda_N g_N(\vec{x})$

because level set consideration (on website)

↳ the solutions to $f(\vec{x})$ occur only at CP of $F(\vec{x})$

↳ therefore, only need to solve $\nabla f = 0$ and find min/max values

Solutions: (1,1) max at e; (-1,1) min at -e

- **ex:** Optimize $f(x,y) = xe^y$ where $x^2+y^2=2$

$x^2+y^2-2 = 0 = g(x,y)$

$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$= xe^y - \lambda(x^2+y^2-2)$

$\nabla F = \langle e^y - 2\lambda x,\ xe^y - 2\lambda y,\ -(x^2+y^2-2) \rangle = 0$

$e^y - 2\lambda x = 0$	$xe^y - 2\lambda y = 0$	$x^2+y^2 = +2$
$y = x^2$	Therefore	$x^2 + x^4 - 2 = 0$
$\lambda = e^{y}/x = e^1/2(1)$	(1,1) is a possible	$(x^2+2)^2(x^2-1) = 0$
$\pm \frac{e}{2}$	extreme of F at	$x = 1 \Rightarrow y = 1$
	$\pm e$ and $(\pm 1, 1)$	$x = -1$

dividing by 2λ